Student number ______

PhD Written Qualifying Exam 2011

Orbits

2011 Orbital Mechanics Ph.D. Written Qualifying Examination

Department of Aerospace Engineering and Engineering Mechanics
The University of Texas at Austin

Friday June 10, 2011

Student Number:__________

Instructions/Comments:

- There are five problems. Each problem is worth 20 points for an exam total of 100 points. The exam duration is 180 minutes.
- This is a closed book and notes examination.
- Symbolic manipulating devices, symbolic memory storage devices, or calculators that can be programmed to solve symbolic or numerical equations, or systems of equations, are not permitted. Calculators that can perform basic numerical calculations are permitted.
- Begin and solve each problem on separate sheets of paper. Write clearly and legibly. In your worksheets label every problem and sub-problem (if applicable) whether or not you provide a solution for it.
- Write your student number on every work sheet.

Constants, Nomenclature, and Equations

Symbol	Value	Units	Description
G	6.673×10^{-20}	$km^3/kg/s^2$	Gravitational Constant
$GM_{earth},\ \mu_{earth}$	398600	km^3/s^2	Earth gravitational parameter
$GM_{moon}\ \mu_{moon}$	4903	km^3/s^2	Moon gravitational parameter
a_{moon}	$384,400$	km	semi-major axis of Moon's orbit wrt to Earth
r_{eq_earth}	6378	km	Earth equatorial radius
r_{eq_moon}	1738	km	Moon equatorial radius
$J_{2_{earth}}$	0.001082636	-	Earth degree 2 zonal

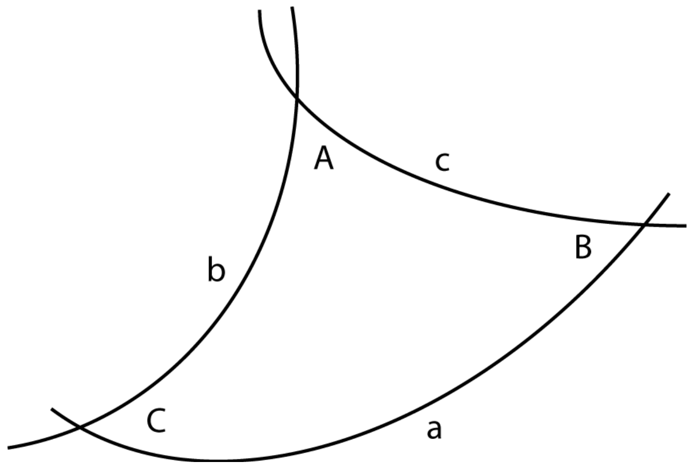
If r is the radius, p is the semi-latus parameter, e is the eccentricity, ν is the true anomaly, a is the semi-major axis, b is the semi-minor axis, n is the mean motion, T_p is the time period, $\mathbf{h}$ is the angular momentum vector, E is the eccentric anomaly, ε is the specific energy, v_∞ is the magnitude of the hyperbolic excess velocity vector, ϕ is the flight path angle, and δ is the turning angle for a non-elliptical orbit, then:

Equation	Description	Condition(s)/Comment
$r = \frac{p}{1+e\cos\nu}$	scalar radius for any conic	$r > 0,\ p > 0, e \geq 0$
$p = \frac{h^2}{\mu}$	semi-latus parameter	$\mu > 0$
$p = a(1-e^2)$	semi-latus parameter	$e < 1$
$b = a\sqrt{1-e^2}$	semi-minor axis	$e < 1$
$n = \sqrt{\frac{\mu}{a^3}}$	mean motion	$e < 1, a > 0$
$T_p = 2\pi\sqrt{\frac{a^3}{\mu}}$	time period	$e < 1, a > 0$
$\mathbf{h} = \mathbf{r} \times \mathbf{v}$	angular momentum vector	
$h = \lvert\mathbf{h}\rvert = rv\cos\phi$	angular momentum magnitude	ϕ is flight path angle
$\mathbf{e} = \frac{1}{\mu}\mathbf{v} \times \mathbf{h} - \frac{\mathbf{r}}{r}$	eccentricity vector	
$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}}$	eccentricity, energy, angular momentum relationship	
$\frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$	true anomaly and eccentric anomaly relationship	$e < 1$
$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$	Specific Energy	
$\varepsilon = -\frac{\mu}{2a}$	Specific Energy	
$\varepsilon = \frac{v_\infty^2}{2}$	Specific Energy	$e > 1$
$\sin\frac{\delta}{2} = \frac{1}{e}$	turning angle and eccentricity relationship	$e \geq 1$
v	magnitude of the velocity vector, $\mathbf{v}$	
r	magnitude of a position vector, $\mathbf{r}$	

Trigonometric Relations

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

Spherical Trigonometric Relations:



$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\begin{aligned}
\cos a &= \cos b \cos c + \sin b \sin c \cos A \\
\cos b &= \cos c \cos a + \sin c \sin a \cos B \\
\cos c &= \cos a \cos b + \sin a \sin b \cos C
\end{aligned}$$

Vector identities:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= \left(\mathbf{a}^\top \mathbf{c}\right) \mathbf{b} - \left(\mathbf{b}^\top \mathbf{c}\right) \mathbf{a} \\
\mathbf{a}^\top (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \times \mathbf{b})^\top \mathbf{c}
\end{aligned}$$

Lagrange's Planetary Equations

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na} \frac{\partial D}{\partial M} \\
\frac{de}{dt} &= \frac{\left(1-e^2\right)^{1/2}}{na^2 e} \left[\left(1-e^2\right)^{1/2} \frac{\partial D}{\partial M} - \frac{\partial D}{\partial \omega} \right] \\
\frac{di}{dt} &= \frac{1}{h \sin i} \left(\cos i \frac{\partial D}{\partial \omega} - \frac{\partial D}{\partial \Omega} \right) \\
\frac{d\Omega}{dt} &= \frac{1}{h \sin i} \frac{\partial D}{\partial i} \\
\frac{d\omega}{dt} &= -\frac{\cos i}{h \sin i} \frac{\partial D}{\partial i} + \frac{\left(1-e^2\right)^{1/2}}{na^2 e} \frac{\partial D}{\partial e} \\
\frac{dM}{dt} &= n - \frac{\left(1-e^2\right)}{na^2 e} \frac{\partial D}{\partial e} - \frac{2}{na} \frac{\partial D}{\partial a}
\end{aligned}$$

where

$$h = na^2 \left(1-e^2\right)^{1/2}$$

Here, n is the mean motion, M is the mean anomaly, i is the inclination, Ω is the right ascension of the ascending node, ω is the argument of periapsis, and D is the scalar disturbing function.

For an orbit about a celestial body with J_2 as the only perturbation, it can be shown that the following orbital elements exhibit a secular variation given by:

$$\begin{aligned}
\dot{\Omega} &= -\frac{3}{2} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i \\
\dot{\omega} &= \frac{3}{4} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \left(5\cos^2 i - 1\right) \\
\dot{M} &= \sqrt{\frac{\mu}{a^3}} + \frac{3}{4} J_2 \frac{n}{\left(1-e^2\right)^{3/2}} \left(\frac{a_e}{a}\right)^2 \left(3\cos^2 i - 1\right)
\end{aligned}$$

Problem 1 (20 points): A new observational tool has located and observed the motion of two celestial bodies that are far removed from any other celestial bodies. Advanced analysis techniques have established that the bodies are essentially spherical with the following gravitational parameters:

$$Gm_1 = 1.5 \times 10^8 \text{ km}^3/\text{sec}^2$$
$$Gm_2 = 0.5 \times 10^8 \text{ km}^3/\text{sec}^2$$

A nonrotating, inertial coordinate system with axes X, Y, Z, has been established, including a specified direction for the X-axis (which points towards a distant star); furthermore, at time t_{ref}, the state of the two bodies is

$$\mathbf{r}_1(t_{ref}) = \begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} -25000 \\ 0 \\ 0 \end{pmatrix} \text{km} \qquad \mathbf{v}_1(t_{ref}) = \begin{pmatrix} \dot{X}_1 \\ \dot{Y}_1 \\ \dot{Z}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 13.693 \\ 0 \end{pmatrix} \text{km/sec}$$

$$\mathbf{r}_2(t_{ref}) = \begin{pmatrix} X_2 \\ Y_2 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 75000 \\ 0 \\ 0 \end{pmatrix} \text{km} \qquad \mathbf{v}_2(t_{ref}) = \begin{pmatrix} \dot{X}_2 \\ \dot{Y}_2 \\ \dot{Z}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -41.079 \\ 0 \end{pmatrix} \text{km/sec}$$

a) (8 points) Sketch the motion of each body in the XYZ frame over some time interval and provide a verbal description of the characteristics of the motion and the relevance of the origin to the dynamical system description.

b) (12 points) Determine the time (with respect to t_{ref}) when the distance of each body with respect to the origin is a minimum.

Problem 2 (20 points):

A spherical near Earth asteroid (NEA) with uniform density, gravitational parameter, $\mu_{nea} = 0.108$ km^3/s^2, and radius $r_{eq\_nea} = 1.0$ km, approaches the Earth on a parabolic path with a periapsis distance of $42,164.197$ km and inclination with respect to the equator of $0°$ degrees. The asteroid skims by an orbiting Earth spacecraft that has an eccentricity, $e^- = 0$, semi-major axis, $a^- = 42,164.197$ km, and inclination, $i^- = 0°$. The problem as posed requires you to examine the effect that a flyby of a NEA has on a satellite in geostationary orbit. The NEA's minimum distance to the spacecraft is 1 km. The inertial frame is centered at the Earth; i.e., the Earth is not affected by the flyby of the NEA. The Earth and the NEA can be approximated as point masses. Use a zero sphere of influence patched conic model to examine the effect of this encounter event on the motion of the satellite. Assume that relative to the NEA, the spacecraft passes the NEA on the near (Earth side) of the NEA.

a: (6 points)

a.1) Sketch the orbits in an Earth centered frame.

a.2) Sketch the velocity flyby diagram of the spacecraft with respect to the NEA

a.3) Sketch the trajectory of the spacecraft in a NEA centered non-rotating frame.

b: (14 points) Compute the flight path angle of the spacecraft after the encounter event, the difference in it's orbital period (in minutes) before and after the encounter event, and its final eccentricity.

Problem 3 (20 points): Consider the following dynamical system with second order equations of motion,

$$\ddot{x} = +2\dot{y} + \frac{\partial U}{\partial x}$$
$$\ddot{y} = -2\dot{x} + \frac{\partial U}{\partial y}$$

where U is a scalar potential function that depends only on the coordinates x, y. The velocities are $\dot{x} = dx/dt, \dot{y} = dy/dt$. Linearize the equations at an arbitrary point of equilibrium to derive the functional form for the first order system

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x}$$

where

$$\delta\mathbf{x} = \begin{pmatrix} \delta x & \delta y & \delta\dot{x} & \delta\dot{y} \end{pmatrix}^\top$$

which represents the deviations of both position and velocity with respect to the point of equilibrium; i.e., if the location of the equilibrium point is given by x_0, y_0 then

$$\delta x = x - x_0$$
$$\delta y = y - y_0$$

a)(2 points) what is the definition of a point of equilibrium for a second order dynamical system with two degrees of freedom?

b)(6 points) what are the elements of the 4×4 system matrix $\mathbf{A}$ for any position dependent scalar potential U as given in the problem statement above? (show the derivation and justification)

$$\mathbf{A} = \begin{pmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{pmatrix}$$

c)(4 points) how can the system matrix $\mathbf{A}$ be used to analyze the linear stability properties associated with a point of equilibrium of this system? (explain briefly in simple words, symbols, and/or equations)

d)(8 points) for a particular dynamical system, and for a particular equilibrium point associated with it, it can be shown that the characteristic equation associated with $\mathbf{A}$ is

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

where μ is a positive parameter that is constrained by $\mu \leq \frac{1}{2}$ and λ is an eigenvalue of $\mathbf{A}$. For what values of μ is perturbed motion considered to be linearly stable at this particular equilibrium point?

Problem 4 (20 points):
Design an Earth satellite orbit to meet the following requirements:

1. Sun-synchronous
2. Perigee fixed at $\omega = 270^\circ$ (in an averaged sense)
3. Orbit period of approximately 3 hours
4. Ground track repeats daily

a)(12 points) Determine a, e, i to satisfy the requirements. Identify (and justify) any assumptions.

b)(8 points) For the resulting perigee and apogee radii values, discuss the impact that these values have on the orbit keeping in mind that the Earth has an atmosphere.

Problem 5 (20 points):

Consider the mean orbital evolution experienced by a satellite orbiting an oblate spheroid (such as the Earth, where oblateness is mostly represented by J_2).

a)(10 points) Derive the expression that is relevant to the disturbing function D associated with J_2 and show that a reasonable representation of the perturbed motion is a "secularly precessing ellipse" for small eccentricity.

b)(10 points) Recall that the right ascension of the ascending node experiences a J_2 dependent change given by

$$\dot{\Omega} = -\frac{3}{2} J_2 \frac{n}{(1-e^2)^2} \left(\frac{a_e}{a}\right)^2 \cos i$$

For small e, where $(1 - e^2) \approx 1$, derive using Lagrange's Planetary equations, the following expression

$$\dot{\Omega} = -\frac{3}{2} n J_2 \left(\frac{a_e}{a}\right)^2 \cos i$$

where a_e is the reference equatorial radius of the body, which for the Earth is r_{eq_earth}.